View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

JOURNAL OF APPROXIMATION THEORY 70, 375–389 (1992)

Approximation by Nörlund Means of Walsh–Fourier Series

F. MÓRICZ*

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, 6720 Szeged, Hungary

AND

A. H. SIDDIQI

Department of Mathematics, Aligarh Muslim University, Aligarh 202002, Uttar Pradesh, India

Communicated by Paul Nevai

Received January 22, 1991; revised August 20, 1991

We study the rate of approximation by Nörlund means for Walsh–Fourier series of a function in L^p and, in particular, in Lip(α, p) over the unit interval $[0, 1)$, where $\alpha > 0$ and $1 \leqslant p \leqslant \infty$. In case $p = \infty$, by L^p we mean C_W, the collection of the uniformly W-continuous functions over $[0, 1)$. As special cases, we obtain the earlier results by Yano, Jastrebova, and Skvorcov on the rate of approximation by Cesàro means. Our basic observation is that the Nörlund kernel is quasi-positive, under fairly general assumptions. This is a consequence of a Sidon type inequality. At the end, we raise two problems. © 1992 Academic Press, Inc.

1. INTRODUCTION

We consider the Walsh orthonormal system $\{w_k(x): k \geqslant 0\}$ defined on the unit interval $I = [0, 1)$ in the Paley enumeration (see [4]). To be more specific, let

$$r_0(x) := \begin{cases} 1 & \text{if} \quad x \in [0, 2^{-1}), \\ -1 & \text{if} \quad x \in [2^{-1}, 1), \end{cases}$$

$$r_0(x+1) := r(x),$$

$$r_j(x) := r_0(2^j x), \qquad j \geqslant 1 \text{ and } x \in I,$$

* This research was conducted while the author was a visiting professor at the Aligarh Muslim University during the spring semester of 1990.

0021-9045/92 $5.00

Copyright © 1992 by Academic Press, Inc.
All rights of reproduction in any form reserved.

be the well-known Rademacher functions. For $k=0$ set $w_0(x)=1$, and if

$$k := \sum_{j=0}^{\infty} k_j 2^j, \qquad k_j = 0 \text{ or } 1,$$

is the dyadic representation of an integer $k \geqslant 1$, then set

$$w_k(x) := \prod_{j=0}^{\infty} [r_j(x)]^{k_j}. \tag{1.1}$$

We denote by $\mathscr{P}_n$ the collection of Walsh polynomials of order less than n, that is, functions of the form

$$P(x) := \sum_{k=0}^{n-1} a_k w_k(x),$$

where $n \geqslant 1$ and $\{a_k\}$ is any sequence of real (or complex) numbers.

Denote by Σ_m the finite σ-algebra generated by the collection of dyadic intervals of the form

$$I_m(k) := [k2^{-m}, (k+1)\, 2^{-m}), \qquad k = 0, 1, ..., 2^m - 1,$$

where $m \geqslant 0$. It is not difficult to see that the collection of Σ_m-measurable functions on I coincides with $\mathscr{P}_{2^m}$, $m \geqslant 0$.

We will study approximation by means of Walsh polynomials in the norm of $L^p = L^p(I)$, $1 \leqslant p < \infty$, and $C_W = C_W(I)$. We remind the reader that C_W is the collection of functions $f\colon I \to \mathbf{R}$ that are uniformly continuous from the dyadic topology of I to the usual topology of $\mathbf{R}$, or in short, uniformly W-continuous. The dyadic topology is generated by the union of Σ_m for $m = 0, 1, ...$.

As is known (see, e.g., [6, p. 9]), a function belongs to C_W if and only if it is continuous at every dyadic irrational of I, is continuous from the right on I, and has a finite limit from the left on $(0, 1]$, all these in the usual topology. Hence it follows immediately that if the periodic extension of a function f from I to $\mathbf{R}$ with period 1 is classically continuous, then f is also uniformly W-continuous on I. The converse statement is not true. For example, the Walsh functions w_k belong to C_W, but they are not classically continuous for $k \geqslant 1$.

For the sake of brevity in notation, we agree to write L^∞ instead of C_W and set

$$\|f\|_p := \left\{ \int_0^1 |f(x)|^p \, dx \right\}^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$\|f\|_\infty := \sup\{|f(x)| : x \in I\}.$$

After these preliminaries, the best approximation of a function $f \in L^p$, $1 \leqslant p \leqslant \infty$, by polynomials in $\mathscr{P}_n$ is defined by

$$E_n(f, L^p) := \inf_{P \in \mathscr{P}_n} \|f - P\|_p.$$

Since $\mathscr{P}_n$ is a finite dimensional subspace of L^p for any $1 \leqslant p \leqslant \infty$, this infimum is attained.

From the results of [6, pp. 142 and 156–158] it follows that L^p is the closure of the Walsh polynomials when using the norm $\|\cdot\|_p$, $1 \leqslant p \leqslant \infty$. In particular, C_W is the uniform closure of the Walsh polynomials.

Next, define the modulus of continuity in L^p, $1 \leqslant p \leqslant \infty$, of a function $f \in L^p$ by

$$\omega_p(f, \delta) := \sup_{|t| < \delta} \|\tau_t f - f\|_p, \qquad \delta > 0,$$

where τ_t means dyadic translation by t:

$$\tau_t f(x) := f(x \dotplus t), \qquad x, t \in I.$$

Finally, for each $\alpha > 0$, Lipschitz classes in L^p are defined by

$$\operatorname{Lip}(\alpha, p) := \{f \in L^p : \omega_p(f, \delta) = \mathcal{O}(\delta^\alpha) \text{ as } \delta \to 0\}.$$

Unlike the classical case, $\operatorname{Lip}(\alpha, p)$ is not trivial when $\alpha > 1$. For example, the function $f := w_0 + w_1$ belongs to $\operatorname{Lip}(\alpha, p)$ for all $\alpha > 0$ since

$$\omega_p(f, \delta) = 0 \qquad \text{when} \quad 0 < \delta < 2^{-1}.$$

2. Main Results

Given a function $f \in L^1$, its Walsh–Fourier series is defined by

$$\sum_{k=0}^{\infty} a_k w_k(x), \qquad \text{where} \quad a_k := \int_0^1 f(t)\, w_k(t)\, dt. \tag{2.1}$$

The nth partial sums of series in (2.1) are

$$s_n(f, x) := \sum_{k=0}^{n-1} a_k w_k(x), \qquad n \geqslant 1.$$

As is well known,

$$s_n(f, x) = \int_0^1 f(x \dotplus t)\, D_n(t)\, dt,$$

where

$$D_n(t) := \sum_{k=0}^{n-1} w_k(t), \qquad n \geqslant 1,$$

is the Walsh–Dirichlet kernel of order n.

Let $\{q_k : k \geqslant 0\}$ be a sequence of nonnegative numbers. The Nörlund means for series (2.1) are defined by

$$t_n(f, x) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} s_k(f, x),$$

where

$$Q_n := \sum_{k=0}^{n-1} q_k, \qquad n \geqslant 1.$$

We always assume that $q_0 > 0$ and

$$\lim_{n \to \infty} Q_n = \infty. \tag{2.2}$$

In this case, the summability method generated by $\{q_k\}$ is regular if and only if

$$\lim_{n \to \infty} \frac{q_{n-1}}{Q_n} = 0. \tag{2.3}$$

As to this notion and result, we refer the reader to [2, pp. 37–38].

We note that in the particular case when $q_k = 1$ for all k, these $t_n(f, x)$ are the first arithmetic or $(C, 1)$-means. More generally, when

$$q_k = A_k^\beta := \binom{\beta + k}{k} \qquad \text{for} \quad k \geqslant 1 \text{ and } q_0 = A_0^\beta := 1,$$

where $\beta \neq -1, -2, \ldots$, the $t_n(f, x)$ are the (C, β)-means for series (2.1).

The representation

$$t_n(f, x) = \int_0^1 f(x \dotplus t) \, L_n(t) \, dt \tag{2.4}$$

plays a central role in the sequel, where

$$L_n(t) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k(t), \qquad n \geqslant 1, \tag{2.5}$$

is the so-called Nörlund kernel.

Our main results read as follow.

THEOREM 1. *Let $f \in L^p$, $1 \leqslant p \leqslant \infty$, let $n = 2^m + k$, $1 \leqslant k \leqslant 2^m$, $m \geqslant 1$, and let $\{q_k : k \geqslant 0\}$ be a sequence of nonnegative numbers such that*

$$\frac{n^{\gamma-1}}{Q_n^\gamma} \sum_{k=0}^{n-1} q_k^\gamma = \mathcal{O}(1) \qquad \text{for some} \quad 1 < \gamma \leqslant 2. \tag{2.6}$$

If $\{q_k\}$ is nondecreasing, then

$$\|t_n(f) - f\|_p \leqslant \frac{5}{2Q_n} \sum_{j=0}^{m-1} 2^j q_{n-2^j} \omega_p(f, 2^{-j}) + \mathcal{O}\{\omega_p(f, 2^{-m})\}, \tag{2.7}$$

while if $\{q_k\}$ is nonincreasing, then

$$\|t_n(f) - f\|_p \leqslant \frac{5}{2Q_n} \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1})\, \omega_p(f, 2^{-j}) + \mathcal{O}\{\omega_p(f, 2^{-m})\}. \tag{2.8}$$

Clearly, condition (2.6) implies (2.2) and (2.3).

We note that if $\{q_k\}$ is nondecreasing, in sign $q_k \uparrow$, then

$$\frac{nq_{n-1}}{Q_n} = \mathcal{O}(1) \tag{2.9}$$

is a sufficient condition for (2.6). In particular, (2.9) is satisfied if

$$q_k \asymp k^\beta \text{ or } (\log k)^\beta \qquad \text{for some} \quad \beta > 0.$$

Here and in the sequel, $q_k \asymp r_k$ means that the two sequences $\{q_k\}$ and $\{r_k\}$ have the same order of magnitude; that is, there exist two positive constants C_1 and C_2 such that

$$C_1 r_k \leqslant q_k \leqslant C_2 r_k \qquad \text{for all } k \text{ large enough.}$$

If $\{q_k\}$ is nonincreasing, in sign $q_k \downarrow$, then condition (2.6) is satisfied if, for example,

$$\begin{aligned} &\text{(i)} \quad q_k \asymp k^{-\beta} && \text{for some} \quad 0 < \beta < 1, \text{ or} \\ &\text{(ii)} \quad q_k \asymp (\log k)^{-\beta} && \text{for some} \quad 0 < \beta. \end{aligned} \tag{2.10}$$

Namely, it is enough to choose $1 < \gamma < \min(2, \beta^{-1})$ in case (i), and $\gamma = 2$ in case (ii).

THEOREM 2. *Let $\{q_k : k \geqslant 0\}$ be a sequence of nonnegative numbers such that in case $q_k \uparrow$ condition* (2.9) *is satisfied, while in case $q_k \downarrow$ condition* (2.10) *is satisfied. If $f \in \operatorname{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leqslant p \leqslant \infty$, then*

$$\|t_n(f) - f\|_p = \begin{cases} \mathcal{O}(n^{-\alpha}) & \text{if } 0 < \alpha < 1, \\ \mathcal{O}(n^{-1} \log n) & \text{if } \alpha = 1, \\ \mathcal{O}(n^{-1}) & \text{if } \alpha > 1. \end{cases} \tag{2.11}$$

Now we make a few historical comments. The rate of convergence of (C, β)-means for functions in $\operatorname{Lip}(\alpha, p)$ was first studied by Yano [10] in the cases when $0 < \alpha < 1$, $\beta > \alpha$, and $1 \leqslant p \leqslant \infty$; then by Jastrebova [1] in the case when $\alpha = \beta = 1$ and $p = \infty$. Later on, Skvorcov [7] showed that these estimates hold for $0 < \beta \leqslant \alpha$ as well, and also studied the cases when $\alpha = 1$, $\beta > 0$, and $1 \leqslant p \leqslant \infty$. In their proofs, the above authors rely heavily on the specific properties of the binomial coefficients A_k^β.

Watari [8] proved that a function $f \in L^p$ belongs to $\operatorname{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leqslant p \leqslant \infty$ if and only if

$$E_n(f, L^p) = \mathcal{O}(n^{-\alpha}).$$

Thus, for $0 < \alpha < 1$ the rate of approximation to functions f in $\operatorname{Lip}(\alpha, p)$ by $t_n(f)$ is as good as the best approximation.

3. AUXILIARY RESULTS

Yano [9] proved that the Walsh–Fejér kernel

$$K_n(t) := \frac{1}{n} \sum_{k=1}^{n} D_k(t) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) w_k(t), \qquad n \geqslant 1,$$

is quasi-positive, and $K_{2^m}(t)$ is even positive. These facts are formulated in the following

LEMMA 1. *Let $m \geqslant 0$ and $n \geqslant 1$; then $K_{2^m}(t) \geqslant 0$ for all $t \in I$,*

$$\int_0^1 |K_n(t)|\, dt \leqslant 2 \qquad \textit{and} \qquad \int_0^1 K_{2^m}(t)\, dt = 1.$$

A Sidon type inequality proved by Schipp and the author (see [3]) implies that the Nörlund kernel $L_n(t)$ is also quasi-positive. More exactly, $C = [\mathcal{O}(1)]^{1/\gamma} 2\gamma/(\gamma - 1)$ in the next lemma, where $\mathcal{O}(1)$ is from (2.6).

LEMMA 2. *If condition* (2.6) *is satisfied, then there exists a constant* C *such that*

$$\int_0^1 |L_n(t)|\, dt \leqslant C, \qquad n \geqslant 1.$$

Now, we give a specific representation of $L_n(t)$, interesting in itself.

LEMMA 3. *Let* $n = 2^m + k$, $1 \leqslant k \leqslant 2^m$, *and* $m \geqslant 1$; *then*

$$\begin{aligned} Q_n L_n(t) = &- \sum_{j=0}^{m-1} r_j(t)\, w_{2^j-1}(t) \sum_{i=1}^{2^j-1} i(q_{n-2^{j+1}+i} - q_{n-2^{j+1}+i+1})\, K_i(t) \\ &- \sum_{j=0}^{m-1} r_j(t)\, w_{2^j-1}(t)\, 2^j q_{n-2^j} K_{2^j}(t) \\ &+ \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1})\, D_{2^{j+1}}(t) \\ &+ Q_{k+1} D_{2^m}(t) + Q_k r_m(t)\, L_k(t). \end{aligned} \tag{3.1}$$

Proof. The technique applied in the proof is essentially due to Skvorcov [7]. By (2.5),

$$\begin{aligned} Q_n L_n(t) &= \sum_{i=1}^{2^m-1} q_{n-i} D_i(t) + q_{n-2^m} D_{2^m}(t) + \sum_{i=2^m+1}^{2^m+k} q_{n-i} D_i(t) \\ &= \sum_{j=0}^{m-1} \sum_{i=0}^{2^j-1} q_{n-2^j-i}(D_{2^j+i}(t) - D_{2^{j+1}}(t)) \\ &\quad + \sum_{j=0}^{m-1} \left(\sum_{i=0}^{2^j-1} q_{n-2^j-i} \right) D_{2^{j+1}}(t) \\ &\quad + q_{n-2^m} D_{2^m}(t) + \sum_{i=1}^{k} q_{n-2^m-i} D_{2^m+i}(t). \end{aligned} \tag{3.2}$$

As is well known (see, e.g., [6, p. 46]),

$$D_{2^m+i}(t) = D_{2^m}(t) + r_m(t)\, D_i(t), \qquad 1 \leqslant i \leqslant 2^m. \tag{3.3}$$

Furthermore, by (1.1), it is not difficult to see that

$$w_{2^j-1-l}(t) = w_{2^j-1}(t)\, w_l(t), \qquad 0 \leqslant l < 2^j.$$

Hence we deduce that

$$D_{2^{j+1}}(t)-D_{2^j+i}(t)=r_j(t)\sum_{l=i}^{2^j-1} w_l(t)=r_j(t)\sum_{l=0}^{2^j-i-1} w_{2^j-1-l}(t)$$
$$=r_j(t)\,w_{2^j-1}(t)\,D_{2^j-i}(t), \qquad 0\leqslant i<2^j. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) yields

$$\begin{aligned}Q_nL_n(t)= &-\sum_{j=0}^{m-1} r_j(t)\,w_{2^j-1}(t)\sum_{i=0}^{2^j-1} q_{n-2^j+1}D_{2^j-i}(t)\\ &+\sum_{j=0}^{m-1}(Q_{n-2^j+1}-Q_{n-2^{j+1}+1})\,D_{2^{j+1}}(t)\\ &+Q_{k+1}D_{2^m}(t)+Q_kr_m(t)\,L_k(t).\end{aligned} \tag{3.5}$$

Performing a summation by part gives

$$\begin{aligned}&\sum_{i=0}^{2^j-1} q_{n-2^j-i}D_{2^j-i}(t)\\ &\quad=\sum_{i=1}^{2^j-1} iK_i(t)(q_{n-2^{j+1}+i}-q_{n-2^{j+1}+i+1})+2^jK_{2^j}(t)\,q_{n-2^j}.\end{aligned}$$

Substituting this into (3.5) results in (3.1).

LEMMA 4. *If* $g\in\mathscr{P}_{2^m}$, $f\in L^p$, *where* $m\geqslant 0$ *and* $1\leqslant p\leqslant\infty$, *then for* $1\leqslant p<\infty$

$$\begin{aligned}&\left\{\int_0^1\left|\int_0^1 r_m(t)\,g(t)[f(x\dotplus t)-f(x)]\,dt\right|^p dx\right\}^{1/p}\\ &\qquad\leqslant 2^{-1}\omega_p(f,2^{-m})\int_0^1|g(t)|\,dt,\end{aligned} \tag{3.6}$$

while for $p=\infty$

$$\begin{aligned}&\sup\left\{\left|\int_0^1 r_m(t)\,g(t)[f(x\dotplus t)-f(x)]\,dt\right|: x\in I\right\}\\ &\qquad\leqslant 2^{-1}\omega_\infty(f,2^{-m})\int_0^1|g(t)|\,dt\end{aligned} \tag{3.7}$$

Proof. Since $g\in\mathscr{P}_{2^m}$, it takes a constant value, say $g_m(k)$ on each dyadic interval $I_m(k)$, where $0\leqslant k<2^m$. We observe that if $t\in I_m(k)$ then $t\dotplus 2^{-m-1}\in I_m(k)$.

We will prove (3.6). By Minkowski's inequality in the usual and in the generalized form, we obtain that

$$\left\{\int_0^1 \left|\int_0^1 r_m(t)\, g(t)[f(x \dotplus t)-f(x)]\, dt\right|^p dx\right\}^{1/p}$$

$$=\left\{\int_0^1 \left|\sum_{k=0}^{2^m-1} g_m(k) \int_{I_{m+1}(2k)} [f(x \dotplus t)-f(x \dotplus t \dotplus 2^{-m-1})]\, dt\right|^p dx\right\}^{1/p}$$

$$\leqslant \sum_{k=0}^{2^m-1} |g_m(k)| \left\{\int_0^1 \left[\int_{I_{m+1}(2k)} |f(x \dotplus t)-f(x \dotplus t \dotplus 2^{-m-1})|\, dt\right]^p dx\right\}^{1/p}$$

$$\leqslant \sum_{k=0}^{2^m-1} |g_m(k)| \int_{I_{m+1}(2k)} \left\{\int_0^1 |f(x \dotplus t)-f(x \dotplus t \dotplus 2^{-m-1})|^p\, dx\right\}^{1/p} dt$$

$$\leqslant \sum_{k=0}^{2^m-1} |g_m(k)|\, 2^{-m-1}\omega_p(f, 2^{-m}).$$

This is equivalent to (3.6).

Inequality to (3.7) can be proved analogously.

4. Proofs of Theorems 1 and 2

We carry out the *proof of Theorem* 1 for $1 \leqslant p < \infty$. The proof for $p = \infty$ is similar and even simpler.

By (2.4), (3.1), and the usual Minkowski inequality, we may write that

$$\begin{aligned}
Q_n \|t_n(f)-f\|_p &:= \left\{\int_0^1 \left|\int_0^1 Q_n L_n(t)[f(x \dotplus t)-f(x)]\, dt\right|^p dx\right\}^{1/p} \\
&\leqslant \sum_{j=0}^{m-1} \left\{\int_0^1 \left|\int_0^1 r_j(t)\, g_j(t)[f(x \dotplus t)-f(x)]\, dt\right|^p dx\right\}^{1/p} \\
&\quad + \sum_{j=0}^{m-1} \left\{\int_0^1 \left|\int_0^1 r_j(t)\, h_j(t)[f(x \dotplus t)-f(x)]\, dt\right|^p dx\right\}^{1/p} \\
&\quad + \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1}) \\
&\quad \times \left\{\int_0^1 \left|\int_0^1 D_{2^{j+1}}(t)[f(x \dotplus t)-f(x)]\, dt\right|^p dx\right\}^{1/p} \\
&\quad + Q_{k+1} \left\{\int_0^1 \left|\int_0^1 D_{2^m}(t)[f(x \dotplus t)-f(x)]\, dt\right|^p dx\right\}^{1/p} \\
&\quad + Q_k \left\{\int_0^1 \left|\int_0^1 r_m(t)\, L_k(t)[f(x \dotplus t)-f(x)]\, dt\right|^p dx\right\}^{1/p} \\
&=: A_{1n} + A_{2n} + A_{3n} + A_{4n} + A_{5n},
\end{aligned} \tag{4.1}$$

say, where

$$g_j(t) := w_{2^j-1}(t) \sum_{i=1}^{2^j-1} i(q_{n-2^{j-1}+i} - q_{n-2^{j+1}+i+1}) K_i(t),$$

$$h_j(t) := w_{2^j-1}(t)\, 2^j q_{n-2^j} q_{n-2^j} K_{2^j}(t), \qquad 0 \leqslant j < m.$$

Applying Lemma 1, in the case when $q_k \uparrow$ we get that

$$\int_0^1 |g_j(t)|\, dt \leqslant 2 \sum_{i=1}^{2^j-1} i |q_{n-2^{j+1}+i} - q_{n-2^{j+1}+i+1}|$$

$$= 2\left(2^j q_{n-2^j} - \sum_{i=1}^{2^j} q_{n-2^{j+1}+i}\right) \leqslant 2^{j+1} q_{n-2^j},$$

while in the case when $q_k \downarrow$

$$\int_0^1 |g_j(t)|\, dt \leqslant 2\left(\sum_{i=1}^{2^j} q_{n-2^{j+1}+i} - 2^j q_{n-2^j}\right)$$

$$\leqslant 2(Q_{n-2^j+1} - Q_{n-2^{j+1}+1}).$$

Thus, by Lemma 4, in the case $q_k \uparrow$

$$A_{1n} \leqslant \sum_{j=0}^{m-1} 2^j q_{n-2^j} \omega_p(f, 2^{-j}), \tag{4.2}$$

while in the case $q_k \downarrow$

$$A_{1n} \leqslant \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1})\, \omega_p(f, 2^{-j}). \tag{4.3}$$

By virtue of Lemmas 1 and 4 again, we obtain that

$$A_{2n} \leqslant 2^{-1} \sum_{j=0}^{m-1} 2^j q_{n-2^j} \omega_p(f, 2^{-j}). \tag{4.4}$$

Obviously, in the case $q_k \downarrow$

$$2^j q_{n-2^j} \leqslant Q_{n-2^j+1} - Q_{n-2^{j+1}+1}. \tag{4.5}$$

Since

$$D_{2^m}(t) = \begin{cases} 2^m & \text{if} \quad t \in [0, 2^{-m}), \\ 0 & \text{if} \quad t \in [2^{-m}, 1) \end{cases}$$

(see, e.g., [6, p. 7]), by the generalized Minkowski inequality, we find that

$$A_{3n} \leqslant \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1}) \times \int_0^1 D_{2^{j+1}}(t) \left\{ \int_0^1 |f(x \dotplus t) - f(x)|^p dx \right\}^{1/p} dt$$

$$\leqslant \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1})\, \omega_p(f, 2^{-j}), \tag{4.6}$$

$$A_{4n} \leqslant Q_{k+1} \omega(f, 2^{-m}). \tag{4.7}$$

Clearly, in the case $q_k \uparrow$

$$Q_{n-2^j+1} - Q_{n-2^{j+1}+1} \leqslant 2^j q_{n-2^j}. \tag{4.8}$$

Finally, by Lemmas 2 and 4, in a similar way to the above we deduce that

$$A_{5n} \leqslant 2^{-1} Q_k \omega_p(f, 2^{-m}) \int_0^1 |L_k(t)|\, dt \leqslant C Q_n \omega_p(f, 2^{-m}). \tag{4.9}$$

Combining (4.1)–(4.9) yields (2.7) in the case $q_k \uparrow$ and (2.8) in the case $q_k \downarrow$.

Proof of Theorem 2. *Case* (a). $q_k \uparrow$. We have

$$n - 2^j \geqslant 2^{m-1} \qquad \text{for} \quad 0 \leqslant j \leqslant m-1.$$

Consequently, for such j's

$$\frac{2^j q_{n-2^j}}{Q_n} = \frac{(n-2^j+1)\, q_{n-2^j}}{Q_{n-2^j+1}} \frac{Q_{n-2^j+1}}{Q_n} \frac{2^j}{n-2^j+1} \leqslant C 2^{j-m+1},$$

where C equals $\mathcal{O}(1)$ from (2.9). Since $f \in \mathrm{Lip}(\alpha, p)$, from (2.7) it follows that

$$\|t_n(f) - f\|_p = \frac{\mathcal{O}(1)}{Q_n} \sum_{j=0}^{m-1} 2^j q_{n-2^j} 2^{-j\alpha} + \mathcal{O}(2^{-m\alpha})$$

$$= \mathcal{O}(1)\, 2^{-m} \sum_{j=0}^{m} 2^{j - \alpha)}$$

$$= \begin{cases} \mathcal{O}(2^{-m\alpha}) & \text{if} \quad 0 < \alpha < 1, \\ \mathcal{O}(m 2^{-m}) & \text{if} \quad \alpha = 1, \\ \mathcal{O}(2^{-m}) & \text{if} \quad \alpha > 1. \end{cases}$$

This is equivalent to (2.11).

Case (b). $q_k \downarrow$. For example, we consider case (i) in (2.10). Then $Q_n \asymp n^{1-\beta}$. This time we have

$$n - 2^{j+1} \geqslant 2^{m-1} \qquad \text{for} \quad 0 \leqslant j \leqslant m-2.$$

Since $f \in \mathrm{Lip}(\alpha, p)$, from (2.8) it follows that

$$\begin{aligned}
\|t_n(f) - f\|_p &\leqslant \frac{5}{2Q_n} \sum_{j=0}^{m-2} 2^j q_{n-2^{j+1}} \omega_p(f, 2^{-j}) \\
&\quad + \frac{5}{2} \omega_p(f, 2^{-m}) + \mathcal{O}\{\omega_p(f, 2^{-m})\} \\
&= \frac{O(1)}{Q_n} \sum_{j=0}^{m-2} 2^j q_{n-2^{j+1}} 2^{-j\alpha} + \mathcal{O}(2^{-m\alpha}) \\
&= \frac{\mathcal{O}(1)\, 2^{-m\beta}}{n^{1-\beta}} \sum_{j=0}^{m-2} 2^{j(1-\alpha)} + \mathcal{O}(2^{-m\alpha}) \\
&= \begin{cases} \mathcal{O}(n^{-1} 2^{m(1-\alpha)}) & \text{if } \ 0 < \alpha < 1, \\ \mathcal{O}(n^{-1} m) & \text{if } \ \alpha = 1, \\ \mathcal{O}(n^{-1}) & \text{if } \ \alpha > 1. \end{cases}
\end{aligned}$$

Clearly, this is equivalent to (2.11).

Case (ii) in (2.10) can be proved analogously.

5. Concluding Remarks and Problems

(A) We have seen that condition (2.6) is satisfied when $q_k = (k+1)^\beta$ for some $\beta > -1$, and Theorems 1 and 2 apply. If q_k increases faster than a positive power of k, then relation (2.6) is no longer true in general. But the case, for example, when q_k grows exponentially is not interesting, since then condition (2.3) of regularity is not satisfied. On the other hand, the case when $\beta = -1$ is of special interest.

Problem 1. Find substitutes of (2.8) and (2.11) when $q_k = (k+1)^{-1}$. In this case, the $t_n(f)$ are called the logarithmic means for series (2.1).

(B) It is also of interest that Theorems 1 and 2 remain valid when

$$q_k \asymp k^\beta \varphi(k), \tag{5.1}$$

where $\beta > -1$ and $\varphi(k)$ is a positive and monotone (nondecreasing or nonincreasing) functions in k, slowly varying in the sense that

$$\lim_{k \to \infty} \frac{\varphi(2k)}{\varphi(k)} = 1.$$

It is not difficult to check that in this case

$$Q_n \asymp n^{1+\beta}\varphi(n).$$

(C) Now, we turn to the so-called saturation problem concerning the Nörlund means $t_n(f)$. We begin with the observation that the rate of approximation by $t_n(f)$ to functions in $\operatorname{Lip}(\alpha, p)$ cannot be improved too much as α increases beyond 1. Indeed, the following is true.

THEOREM 3. *If $\{q_k\}$ is a sequence of nonnegative numbers such that*

$$\liminf_{m \to \infty} q_{2^m - 1} > 0, \tag{5.2}$$

and if for some $f \in L^p$, $1 \leqslant p \leqslant \infty$,

$$\|t_{2^m}(f) - f\|_p = o(Q_{2^m}^{-1}) \qquad as \quad m \to \infty, \tag{5.3}$$

then f is constant.

We note that condition (5.2) is certainly satisfied if $q_k \uparrow$ or $q_k \downarrow$ and $\lim q_k > 0$.

Proof. Since by definition

$$E_{2^m}(f, L^p) \leqslant \|t_{2^m}(f) - f\|_p,$$

and by a theorem of Watari [8]

$$\|s_{2^m}(f) - f\|_p \leqslant 2E_{2^m}(f, L^p),$$

it follows from (5.3) that

$$\|s_{2^m}(f) - f\|_p = o(Q_{2^m}^{-1}) \qquad \text{as} \quad m \to \infty. \tag{5.4}$$

A simple computation gives that

$$Q_{2^m}\{s_{2^m}(f, x) - t_{2^m}(f, x)\} = \sum_{k=1}^{2^m - 1} (Q_{2^m} - Q_{2^m - k})\, a_k w_k(x).$$

Now, (5.3) and (5.4) imply that

$$\lim_{m \to \infty} \left\| \sum_{k=1}^{2^m - 1} (Q_{2^m} - Q_{2^m - k})\, a_k w_k(x) \right\|_p = 0.$$

Since $\|\cdot\|_1 \leqslant \|\cdot\|_p$, for any $p \geqslant 1$ it follows that

$$\begin{aligned}
&\lim_{m\to\infty} |(Q_{2^m} - Q_{2^m-j})\, a_j| \\
&\quad = \lim_{m\to\infty} \left| \int_0^1 w_j(x) \left\{ \sum_{k=1}^{2^m-1} (Q_{2^m} - Q_{2^m-k})\, a_k w_k(x) \right\} dx \right| \\
&\quad \leqslant \lim_{m\to\infty} \left\| \sum_{k=1}^{2^m-1} (Q_{2^m} - Q_{2^m-k})\, a_k w_k(w) \right\|_1 = 0.
\end{aligned}$$

Hence, by (5.2), we conclude that $a_j = 0$ for all $j \geqslant 1$. Therefore, $f = a_0$ is constant.

In the particular case when $q_k = 1$ for all k, the $t_n(f)$ are the $(C, 1)$-means for series (2.1) defined by

$$\sigma_n(f, x) := \frac{1}{n} \sum_{k=1}^{n} s_k(f, x), \qquad n \geqslant 1,$$

and Theorem 3 is known (see, e.g., [6, p. 191]). It says that if for some $f \in L^p$, $1 \leqslant p \leqslant \infty$,

$$\|\sigma_{2^m}(f) - f\|_p = o(2^{-m}) \qquad \text{as} \quad m \to \infty,$$

then f is necessarily constant.

Problem 2. How can one characterize those functions $f \in L^p$ such that

$$\|\sigma_n(f) - f\|_p = \mathcal{O}(n^{-1}) \qquad \text{for some} \quad 1 \leqslant p \leqslant \infty? \tag{5.5}$$

We conjecture that (5.5) holds if and only if

$$\sum_{m=0}^{\infty} 2^m \omega_p(f, 2^{-m}) < \infty, \qquad \text{or equivalently} \qquad \sum_{k=1}^{\infty} \omega_p(k^{-1}) < \infty.$$

The "if" part can be proved in the same manner as in the case when $\omega_p(f, \delta) = \mathcal{O}(\delta^\alpha)$ for some $\alpha > 1$ (cf. [6, p. 190]). The proof (or disproof) of the "only if" part is a problem.

(D) Finally, we note that the results of this paper can be carried over to the systems that are obtained from the Walsh–Paley system $\{w_k(x)\}$ by means of the so-called piecewise linear rearrangements introduced by Schipp [5]. (See also [7].) In particular, the Walsh–Kaczmarz system is among them.

References

1. M. A. Jastrebova, On approximation of functions satisfying the Lipschitz condition by arithmetic means of their Walsh–Fourier series, *Mat. Sb.* **71** (1966), 214–226. [Russian]
2. C. N. Moore, "Summable Series and Convergence Factors," American Mathematical Society Colloquium Publications, Vol. 22, Amer. Math. Soc., Providence, RI, 1938.
3. F. Móricz and F. Schipp, On the integrability and L^1-convergence of Walsh series with coefficients of bounded variation, *J. Math. Anal. Appl.* **146** (1990), 99–109.
4. R. E. A. C. Paley, A remarkable system of orthogonal functions, *Proc. London Math. Soc.* **34** (1932), 241–279.
5. F. Schipp, On certain rearrangements of series with respect to the Walsh system, *Mat. Zametki* **18** (1975), 193–201. [Russian]
6. F. Schipp, W. R. Wade, and P. Simon, "Walsh Series. An Introduction to Dyadic Harmonic Analysis," Akadémiai Kiadó, Budapest, 1990.
7. V. A. Skvorcov, Certain estimates of approximation of functions by Cesàro means of Walsh–Fourier series, *Mat. Zametki* **29** (1981), 539–547. [Russian]
8. C. Watari, Best approximation by Walsh polynomials, *Tôhoku Math. J.* **15** (1963), 1–5.
9. Sh. Yano, On Walsh series, *Tôhoku Math. J.* **3** (1951), 223–242.
10. Sh. Yano, On approximation by Walsh functions, *Proc. Amer. Math. Soc.* **2** (1951), 962–967.